MATH 312
Lecture 26

THE SIMPLEX Method

Recap:
- Linear INequalities : $A\vec{x} \le \vec{b}$
- Geometric view as half-space intersections
- Feasible regions: bounded, unbounded or empty
- Optimization of linear functions over feasible regions is LINEAR PROGRAMMING: <u>optima occur at corners</u>.

→ STANDARD / PRIMAL FORM: given $A, \vec{b}, \vec{c}$

Maximize $c^T x$ subject to $Ax \le b$ and $x \ge 0$

→ <u>Dual</u> form:

Minimize $b^T p$ subject to $A^T p \ge c$ and $p \ge 0$.

<u>Thm</u> [Weak duality] If there are solutions to both the primal and dual problems (call them x_* and p_*) then:
$$\boxed{c^T x_* \le b^T y_*}$$

<u>The</u> <u>POWER</u> <u>of</u> <u>Duality</u>: sometimes the dual problem is MUCH easier to solve than the primal (or vice-versa).
This typically occurs when either
1. there are more variables than constraints, or
2. there are more constraints than variables.

Eg: Maximize $3x + 2y$ subject to $x + y \le 5$ and $x, y \ge 0$
vs Minimize $5p$ subject to $p \ge 3$ and $p \ge 0$.
$(p \ge 2)$

Solving the dual <u>already</u> tells us $3x + 2y \le 15$

Q How to solve LARGE linear programming problems when # variables and # constraints are <u>BOTH</u> huge??

A Linear Algebra, of course.

The <u>problem</u> is that $Ax \leq b$ is not attackable by the usual row-operation tricks: after all, the feasible region is not usually a single point or a subspace, or anything "nice".

The <u>solution</u> is SLACK VARIABLES: replace each inequality of the form $ax+by \leq c$ by

$$ax+by+r=c \quad ; \quad r \geq 0$$

where r is our slack variable.

IDEA	Slack variables turn inequalities to equalities!

So, instead of

Max $3x-4y$ sub. to $x+y \leq 10$, $3x+2y \leq 6$ and $x,y \geq 0$

We have the SLACK form

Max $3x-4y$ sub. to $x+y+r=10$, $3x+2y+s=10$ and $x,y,r,s \geq 0$.

And <u>NOW</u> the linear algebra takes over.

SIMPLEX METHOD | Tutorial

The method is all about "inspired" row operations. First, one sets up a suitable "augmented matrix" for the slack problem

$$\text{Max } 3x-4y \quad \text{sub to} \quad \begin{array}{l} x+y+r=10 \\ 3x+2y+s=6 \end{array} \quad \text{and } x,y,r,s \geq 0$$

Here it is: $A=\begin{bmatrix}1&1\\3&2\end{bmatrix}$, $b=\begin{bmatrix}10\\6\end{bmatrix}$ and $c=\begin{bmatrix}3\\-4\end{bmatrix}$

$$\begin{array}{cccccc} ``3x-4y" & x & y & r & s & \text{Value} \\ \end{array}$$

$$\left[\begin{array}{c|cc|cc|c} 1 & \overbrace{-3 \quad 4}^{-c^T} & & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 10 \\ 0 & \underbrace{3 \quad 2}_{A} & & \underbrace{0 \quad 1}_{Id} & & \underbrace{6}_{b} \end{array}\right] = \text{"B"}$$

Let's call this

For a <u>general</u> lin. prog problem "Max $c^T x$ sub to $Ax \leq b$ and $x \geq 0$" this matrix has the <u>BLOCK</u> form:

$$\left[\begin{array}{ccc|c} 1 & -c^T & 0 & 0 \\ 0 & A & Id & b \end{array}\right] = B$$

This initial matrix sets $x, y = 0$ and $r=10$, $s=6$. One can read it as follows:

- Each pivot variable has value in the last column
- Each non-pivot variable equals zero.

Note the linear system being solved here:

$$\begin{array}{ccc} \text{"Ans"} & \text{"x"} & \text{"s"} \\ \end{array}$$
$$\begin{bmatrix} 1 & -c^T & 0 \\ 0 & A & Id \end{bmatrix} \begin{bmatrix} a \\ x \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad \text{i.e.,}$$

Where x are the given variables and s the slack variables. So, we get

$$\boxed{\begin{array}{l} a = c^T x, \\ Ax + Id\, s = b \end{array}}$$

$\to$ (final answer we want)

$\to$ (the equalities which used to be inequalities $Ax \le b$)

Any NEGATIVE entry in the top row is making "a" SMALLER, since the entries in x and s are ≥ 0.

So, here's step 1:

I. Identify a column of B whose first element is NEGATIVE. In our example,

$$B = \left[\begin{array}{ccccc|c} 1 & -3 & 4 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 10 \\ 0 & 1 & 2 & 0 & 1 & 6 \end{array}\right].$$

Only column 2, $\begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$ qualifies!

II Now, we need to decide which of the two ones UNDER the offending "-3" of this column should be selected as the "PIVOT"... 1 or 3?

There's an easy way to do this: look at our chosen column, the LAST column and the corresponding "quotients":

OUR COLUMN	LAST COLUMN	Last col/our col
-3	0	(IGNORE THIS)
1	10	$10/1 = 10$
3	6	$6/3 = 2$

Now pick the row which gives the $\boxed{\text{MINIMUM}}$ of these quotients on the right, which in our case is Row 3 as clearly $2 \leq 10$ ... So, we want to make 3 the PIVOT:

$$B = \left[\begin{array}{ccccc|c} 1 & -3 & 4 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 10 \\ 0 & 3 & 2 & 0 & 1 & 6 \end{array}\right]$$

Want to pivotize!

The row operations which accomplish this are:

$R_1' = R_1 + R_3$ (to kill -3)

$R_2' = R_2 - \frac{1}{3} R_3$ (to kill 1)

$R_3' = R_3/3$ (to make 3 become 1)

So, we get

$$B' = \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 0 & 6 & 0 & 1 & 6 \\ 0 & 0 & -5/3 & 1 & 4/3 & 8 \\ 0 & 1 & -2 & 0 & -1 & 2 \end{array}\right] \\ \begin{array}{cccccc} \text{"Ans"} & x & y & s & t & \end{array} \end{array}$$

At this step, we have:

$\boxed{\text{"Ans = 6"}}$

[pivot variables have values at the rightmost column; free variables (those without pivots) are ZERO]

- $x = 2$; $y = 0$ [given variables]
- $s = 8$; $t = 0$ [slack variables]

We STOP when the top row is positive (as it is now). Otherwise, we REPEAT:

Pick another column with first entry negative, etc.